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Infinite-dimensional Lie algebras in 4D conformal quantum field theory*

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Abstract
The concept of global conformal invariance (GCI) opens the way of applying algebraic techniques, developed in the context of two-dimensional chiral conformal field theory, to a higher (even) dimensional spacetime. In particular, a system of GCI scalar fields of conformal dimension two gives rise to a Lie algebra of harmonic bilocal fields, $V_M(x, y)$, where the M span a finite dimensional real matrix algebra $\mathcal{M}$ closed under transposition. The associative algebra $\mathcal{M}$ is irreducible iff its commutant $\mathcal{M}'$ coincides with one of the three real division rings. The Lie algebra of (the modes of) the bilocal fields is in each case an infinite-dimensional Lie algebra: a central extension of $sp(\infty, \mathbb{R})$ corresponding to the field $\mathbb{R}$ of reals, of $u(\infty, \infty)$ associated with the field $\mathbb{C}$ of complex numbers, and of $so^*(4\infty)$ related to the algebra $\mathbb{H}$ of quaternions. They give rise to quantum field theory models with superselection sectors governed by the (global) gauge groups $O(N)$, $U(N)$ and $U(N, \mathbb{H}) = Sp(2N)$, respectively.

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1. Introduction

The assumption of global conformal invariance—which says that we are dealing with a single valued representation of $SU(2,2)$ rather than with a representation of its covering—in (four-dimensional) Minkowski space has surprisingly strong consequences [18]. Combined with the Wightman axioms, it implies *Huygens locality*, which yields the vertex-algebra-type condition

$$((x-y)^2)^n[\phi(x),\psi(y)]=0 \qquad \text{for} \qquad n\gg 0 \tag{1.1}$$

for any pair ϕ, ψ of local Bose fields ($n \gg 0$ meaning 'n sufficiently large'). Huygens locality and energy positivity imply, in turn, rationality of correlation functions. A GCI quantum field theory (QFT) that admits a stress–energy tensor (something, we here assume) necessarily involves infinitely many conserved symmetric tensor currents in the operator product expansion (OPE) of any Wightman field with its conjugate. The twist two contributions give rise to a harmonic *bifield* $V(x, y)$, which is an important tool in the study of GCI QFT models [16, 17, 14, 2, 15]. The spectacular development of two-dimensional (2D) *conformal field theory* in the 1980's is based on the preceding study of infinite-dimensional (Kac–Moody and Virasoro) Lie algebras and their representations. A straightforward generalization of this tool did not seem to apply in higher dimensions. After the first attempts to construct (4D) Poincaré invariant Lie fields led to examples violating energy positivity [13], it was proven [3], that scalar Lie fields do not exist in three or more dimensions. It is therefore important to realize that the argument does not pass to *bi*fields, and that the above mentioned harmonic bifields do give rise to infinite-dimensional Lie algebras.

Consider bilocal fields of the form

$$V_M(x,y)=\sum_{ij}M_{ij}{:}\varphi_i(x)\varphi_j(y){:}, \tag{1.2}$$

where M is a real matrix and φ_j are a system of independent real massless free fields. According to Wick's theorem, the commutator of $V_{M_1}(x_1,x_2)$ and $V_{M_2}(x_3,x_4)$ is:

$$\begin{aligned}\left[V_{M_1}(x_1,x_2),V_{M_2}(x_3,x_4)\right] &= \Delta_{2,3}V_{M_1M_2}(x_1,x_4)+\Delta_{2,4}V_{M_1{}^tM_2}(x_1,x_3)\\ &\quad+\Delta_{1,3}V_{{}^tM_1M_2}(x_2,x_4)+\Delta_{1,4}V_{{}^tM_1{}^tM_2}(x_2,x_3)\\ &\quad+\mathrm{tr}(M_1M_2)\Delta_{12,34}+\mathrm{tr}({}^tM_1M_2)\Delta_{12,43},\end{aligned} \tag{1.3}$$

where ${}^tM$ is the transposed matrix, $\Delta_{j,k}$ is the free field commutator, $\Delta_{j,k}=\Delta^+_{j,k}-\Delta^+_{k,j}$, and $\Delta_{jk,lm}=\Delta^+_{j,m}\Delta^+_{k,l}-\Delta^+_{m,j}\Delta^+_{l,k}$ for $\Delta^+_{j,k}:=\Delta_+(x_j-x_k)$, the two point massless scalar correlation function.

It is one of the main results of [15] that the same abstract structure can be derived from first principles in GCI quantum field theory. More precisely, the twist two bilocal fields appearing in the OPE of any two scalar fields of dimension two can be linearly labeled by matrices M such that the commutation relations (1.3) hold. From this, the representation (1.2) can be deduced. In the present paper we shall consider only finite size matrices; in general, the system of independent massless free fields can be infinite and then M's should be assumed to be Hilbert–Schmidt operators.

The question arises, whether there are nontrivial linear subspaces $\mathcal{M}$ of real matrix algebras upon which the commutation relations of the corresponding bifields V_M ($M \in \mathcal{M}$) close. We shall call such systems of bifields *Lie systems*, or, *Lie bifields*. It follows from (1.3) that if $\mathcal{M}$ is a *t-subalgebra* (i.e. a subalgebra closed under transposition) of the real matrix algebra, then $\{V_M\}_{M\in\mathcal{M}}$ is a Lie system. Conversely, any Lie system corresponds to a subalgebra $\mathcal{M}$ such that ${}^tM_1M_2$, $M_1{}^tM_2$, ${}^tM_1{}^tM_2 \in \mathcal{M}$ whenever $M_1, M_2 \in \mathcal{M}$. In particular, if $\mathcal{M}$ contains the identity matrix, then it is a t-subalgebra.

2. *t*-subalgebras of real matrix algebras

Let us consider *t*-subalgebras $\mathcal{M}$ of the matrix algebra $Mat(L, \mathbb{R})$, where L is a positive integer (equal to the number of fields φ_j). The classification of all such $\mathcal{M}$ is a classical mathematical problem, which goes back to Frobenius, Schur and Wedderburn (see, e.g. [11, chapter XVII] and [4, chapter 9, appendix II]).

We first observe that $\mathcal{M}$ is equipped with the Frobenius inner product

$$\langle M_1, M_2\rangle = \mathrm{tr}({}^t M_1 M_2) = \sum_{ij} (M_1)_{ij}(M_2)_{ij}, \tag{2.1}$$

which is symmetric, positive definite, and has the property $\langle M_1 M_2, M_3\rangle = \langle M_1, M_3 {}^t M_2\rangle$. This implies that for every right ideal $\mathcal{I} \subset \mathcal{M}$, the orthogonal complement $\mathcal{I}^\perp$ is again a right ideal. Note also that $\mathcal{I}$ is a right ideal if and only if ${}^t\mathcal{I}$ is a left ideal. Therefore, $\mathcal{M}$ is a *semisimple* algebra (i.e. a direct sum of left ideals), and every module over $\mathcal{M}$ is a direct sum of irreducible ones.

Now assume, without loss of generality, that the algebra $\mathcal{M} \subset End_{\mathbb{R}}\mathcal{L} \cong Mat(L, \mathbb{R})$ acts irreducibly on the vector space $\mathcal{L} \cong \mathbb{R}^L$. Let $\mathcal{M}' \subset End_{\mathbb{R}}\mathcal{L}$ be the *commutant* of $\mathcal{M}$, i.e. the set of all matrices M commuting with all elements of $\mathcal{M}$. Then by Schur's lemma (whose real version [11] is much less popular than the complex one), $\mathcal{M}'$ is a real division algebra. By the Frobenius theorem, $\mathcal{M}'$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ as a real algebra (where $\mathbb{H}$ denotes the algebra of *quaternions*). Finally, the classical Wedderburn theorem gives that $\mathcal{M}$ is isomorphic to the matrix algebra $End_{\mathcal{M}'}\mathcal{L}$. In addition, since $\mathcal{M}$ is closed under transposition, then $\mathcal{M}'$ is also a *t*-algebra, and the transposition in $\mathcal{M}'$ coincides with the conjugation in $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively.

Observe that, since $\mathcal{M} \cong End_{\mathbb{F}}\mathcal{L}$ (where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$), we can view $\mathcal{L}$ as a left $\mathbb{F}$-module on which $\mathcal{M}$ acts $\mathbb{F}$-linearly. Alternatively, $\mathcal{L}$ can be made an $(\mathcal{M}, \mathbb{F})$-bimodule by setting $M \cdot f \cdot M' := M({}^t M')f$ for $f \in \mathcal{L}$, $M \in \mathcal{M}$ and $M' \in \mathcal{M}' \cong \mathbb{F}$. Then the embedding $\mathbb{F} \subset End_{\mathbb{F}}\mathcal{L} \cong \mathcal{M}$ endows $\mathcal{L}$ with the structure of an $\mathbb{F}$-bimodule. In other words, we have two commuting copies, left and right, of $\mathbb{F}$ in $End_{\mathbb{R}}\mathcal{L}$, which are subalgebras of $\mathcal{M}$ and $\mathcal{M}'$, respectively. Moreover, denoting $N = dim_{\mathbb{F}}\,\mathcal{L}$, we have: $L = dim_{\mathbb{R}}\,\mathcal{L} = N dim_{\mathbb{R}}\mathbb{F} = N, 2N$ or $4N$ when $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, respectively.

If $\mathcal{M}$ is not an irreducible *t*-subalgebra of $Mat(L, \mathbb{R})$, i.e. $\mathcal{L} \cong \mathbb{R}^L$ is not an irreducible $\mathcal{M}$-module, then $\mathcal{L}$ splits into irreducible submodules, each of them of the above three types

$$\mathcal{L} = \mathcal{L}_{\mathbb{R}} \oplus \mathcal{L}_{\mathbb{C}} \oplus \mathcal{L}_{\mathbb{H}}, \tag{2.2}$$

where each $\mathcal{L}_{\mathbb{F}}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) is an $\mathbb{F}$-module such that $\mathcal{M}$ acts on it $\mathbb{F}$-linearly. In our QFT application, the space $\mathcal{L}$ is the real linear span of the real massless scalar fields φ_j, and then a Lie system of bifields V_M splits into three subsystems: of types $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. The first two cases were considered in a previous paper [2] and led to gauge groups of type $U(N, \mathbb{R}) = O(N)$ and $U(N, \mathbb{C}) = U(N)$, respectively, where $N = dim_{\mathbb{F}}\mathcal{L}$. Here we are going to consider the third case in which, as we shall see, the gauge groups that arise are of type $U(N, \mathbb{H}) = Sp(2N)$, the compact real form of the *symplectic group*.

In each of the three cases, the associated infinite-dimensional Lie algebra (1.3) has a central charge proportional to the order N of the gauge group $U(N, \mathbb{F})$.

3. Irreducible Lie bifields and associated dual pairs

In this section, we consider Lie bifields $\{V_M\}_{M\in\mathcal{M}}$ corresponding to irreducible *t*-subalgebras $\mathcal{M}$ of $Mat(L, \mathbb{R})$. As discussed in the previous section, we have $\mathcal{M} \cong End_{\mathcal{M}'}\mathcal{L}$, where $\mathcal{L} \cong \mathbb{R}^L$ and the commutant $\mathcal{M}' \cong \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

In the case when $\mathcal{M}' \cong \mathbb{R}$ and $dim_{\mathbb{R}}\mathcal{L} = 1$, we have one bifield

$$V(x, y) = :\varphi(x)\varphi(y):. \tag{3.1}$$

More generally, V can be taken a sum of N independent copies of Lie bifields of type (3.1),

$$V(x, y) \equiv V_{(N)}(x, y) = :\boldsymbol{\varphi}(x)\boldsymbol{\varphi}(y): = \sum_{j=1}^{N} :\varphi_j(x)\varphi_j(y):, \tag{3.2}$$

which is invariant under the *gauge group* $O(N)$ (including reflections). Here $L = N$ and $O(N)$ is realized as the group of linear automorphisms of $\mathcal{L} = Span_{\mathbb{R}}\{\varphi_j\}$ preserving the quadratic form (3.2) in φ_j. In this case the *field Lie algebra* (i.e. the Lie algebra of field modes corresponding to the eigenvalues of the one-particle energy, see the appendix) is isomorphic to a central extension of $sp(\infty, \mathbb{R})$ of central charge N; see [2].

The case when $\mathcal{M}' \cong \mathbb{C}$ and $dim_{\mathbb{C}}\mathcal{L} = 1$ is given by two real bifields, $V_{\mathbf{1}}$ and V_ε that correspond to the 2×2 matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.3}$$

They are thus generated by two independent real massless fields $\varphi_1(x)$ and $\varphi_2(x)$:

$$\begin{aligned} V_{\mathbf{1}}(x, y) &= :\varphi_1(x)\varphi_1(y): + :\varphi_2(x)\varphi_2(y):, \\ V_\varepsilon(x, y) &= :\varphi_1(x)\varphi_2(y): - :\varphi_2(x)\varphi_1(y):. \end{aligned} \tag{3.4}$$

Combining φ_1 and φ_2 into one complex field $\boldsymbol{\varphi}(x) = \varphi_1(x) + \mathrm{i}\varphi_2(x)$ we get that $V_{\mathbf{1}}$ and V_ε are the real and the imaginary parts of the complex bifield

$$W(x, y) = :\boldsymbol{\varphi}^*(x)\boldsymbol{\varphi}(y): = V_{\mathbf{1}}(x, y) + \mathrm{i}V_\varepsilon(x, y). \tag{3.5}$$

Taking again N independent copies of such Lie bifields,

$$W_{(N)}(x, y) = \sum_{j=1}^{N} :\boldsymbol{\varphi}_j^*(x)\boldsymbol{\varphi}_j(y):, \qquad \boldsymbol{\varphi}_j(x) = \varphi_{1,j}(x) + \mathrm{i}\varphi_{2,j}(x), \tag{3.6}$$

we get a *gauge group* $U(N)$, where $L = 2N$. The *field Lie algebra* in this second case is isomorphic to a central extension of $u(\infty, \infty)$ again of central charge N ([2]).

Finally, for $\mathcal{M}' = \mathbb{H}$ the minimal size of the matrices in $\mathcal{M}$ is four. We can formally derive the basic bifields V_M in this case as in the above complex case (3.5). Let us combine the four independent scalar fields $\varphi_j(x)$ ($j = 0, 1, 2, 3$) in a single 'quaternionic-valued' field and its conjugate

$$\begin{aligned} \boldsymbol{\varphi}(x) &= \varphi_0(x) + \varphi_1(x)I + \varphi_2(x)J + \varphi_3(x)K, \\ \boldsymbol{\varphi}^+(x) &= \varphi_0(x) - \varphi_1(x)I - \varphi_2(x)J - \varphi_3(x)K, \end{aligned} \tag{3.7}$$

where I, J, K are the (imaginary) quaternionic units satisfying $IJ = K = -JI$, $I^2 = J^2 = K^2 = -1$. This allows us to write a quaternionic bifield Y as

$$Y(x, y) = :\boldsymbol{\varphi}^+(x)\boldsymbol{\varphi}(y): = V_0(x, y) + V_1(x, y)I + V_2(x, y)J + V_3(x, y)K, \tag{3.8}$$

where the components V_α ($\alpha = 0, 1, 2, 3$) of Y can be further expressed in terms of the four-vectors $\boldsymbol{\varphi}$ and a 4×4 matrix realization of the quaternionic units in a manner similar to (3.4)

$$V_\alpha(x,y) \equiv V_{\ell_\alpha}(x,y) = :\varphi(x)\ell_\alpha\varphi(y):,$$
$$\ell_0 = \mathbf{1}, \qquad \ell_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.9}$$
$$\ell_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \ell_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to check that the 4×4 matrices ℓ_α generate the quaternionic algebra $\mathbb{H} \cong \mathcal{M}$. The commutant $\mathcal{M}'$ in $Mat(4,\mathbb{R})$ is spanned by the unit matrix and another realization of the imaginary quaternionic units as a set of real antisymmetric 4×4 matrices $r_k (k = 1, 2, 3)$. The two sets $\{r_k\}_{k=1}^3$ and $\{\ell_k\}_{k=1}^3$ correspond to the splitting of the Lie algebra $so(4)$ into a direct sum of two $so(3)$ algebras

$$\begin{aligned} &\ell_1 = \sigma_3 \otimes \varepsilon, \qquad \ell_2 = \varepsilon \otimes \mathbf{1}, \qquad \ell_3 = \ell_1\ell_2 = \sigma_1 \otimes \varepsilon, \\ &r_1 = \varepsilon \otimes \sigma_3, \qquad r_2 = \mathbf{1} \otimes \varepsilon, \qquad r_3 = r_1 r_2 = -r_2 r_1 = \varepsilon \otimes \sigma_1, \end{aligned} \tag{3.10}$$

where σ_k are the Pauli matrices and $\varepsilon = \mathrm{i}\sigma_2$ as in (3.3).

We shall demonstrate that the quaternionic field Y (3.8) generates a central extension of the Lie algebra[4] $so^*(4\infty)$. To this end, we represent Y by a pair of complex bifields

$$\begin{aligned} W(x,y) &= \tfrac{1}{2}(V_0(x,y) + \mathrm{i}V_3(x,y)) \\ &=: \psi_1^*(x)\psi_1(y) : + : \psi_2^*(x)\psi_2(y) := W(y,x)^*, \\ A(x,y) &= \tfrac{1}{2}(V_1(x,y) - \mathrm{i}V_2(x,y)) \\ &= \psi_1(x)\psi_2(y) - \psi_2(x)\psi_1(y) = -A(y,x), \end{aligned} \tag{3.11}$$

and their conjugates, where ψ_α are complex linear combinations of φ_ν

$$\psi_1 = \frac{1}{\sqrt{2}}(\varphi_0 + \mathrm{i}\varphi_3), \qquad \psi_2 = \frac{1}{\sqrt{2}}(\varphi_1 - \mathrm{i}\varphi_2). \tag{3.12}$$

Substituting as above each φ_ν (respectively ψ_α) by an N-vector of commuting free fields we can write the nontrivial local commutation relations (CR) of $W(1,2) \equiv W(x_1,x_2)$ and $A(1,2)$ in the form

$$[W^*(1,2), W(3,4)] = \Delta_{1,3}W(2,4) + \Delta_{2,4}W^*(1,3) + 2N\Delta_{12,43}; \tag{3.13}$$

$$\begin{aligned} &[W(1,2), A(3,4)] = \Delta_{1,3}A(2,4) - \Delta_{1,4}A(2,3), \\ &[W(1,2), A^*(3,4)] = \Delta_{2,3}A^*(1,4) - \Delta_{2,4}A^*(1,3), \\ &[A^*(1,2), A(3,4)] = \Delta_{1,3}W(2,4) - \Delta_{1,4}W(2,3) + \Delta_{2,4}W(1,3) \\ &\qquad\qquad - \Delta_{2,3}W(1,4) + 2N(\Delta_{12,43} - \Delta_{12,34}). \end{aligned} \tag{3.14}$$

In particular, W coincides with $W_{(2N)}$ in (3.6) and generates the $u(\infty,\infty)$ algebra (of even central charge), which contains the compact Cartan subalgebra of $so^*(4\infty)$; see the appendix. On the other hand, it is straightforward to display the gauge group in the original picture as the invariance group of the quaternionic-valued bifield Y (3.8) viewed as a quaternionic form in

[4] For a description of the Lie algebra $so^*(2n)$ of the noncompact group $SO^*(2n)$ and of its highest weight representations, see [7]. For an oscillator realization of the Lie superalgebra $osp(2m^*|2n)$ (with even subalgebra $so^*(2m) \times sp(2n)$), see [8]. If we view $so^*(4\infty)$ as an inductive limit of $so^*(4n)$ then the central extension is trivial.

the N-dimensional space of real quaternions. We obtain the group of $N \times N$ unitary matrices with quaternionic entries

$$U(N, \mathbb{H}) = Sp(2N) \equiv USp(2N), \tag{3.15}$$

i.e. the compact group of unitary complex symplectic $2N \times 2N$ matrices.

4. Unitary positive-energy representations and superselection structure

Two important developments, one in QFT, the other in representation theory, originated half a century ago from the talks of Rudolf Haag and Irving Segal at the first Lille conference [12] on mathematical problems in QFT. Later they gradually drifted apart and lost sight of each other. The work of the Hamburg–Rome–Göttingen school on the operator algebra approach to local quantum physics [9] culminated in the theory of (global) gauge groups and superselection sectors [6, 5]. The parallel development of the theory of highest weight modules of semisimple Lie groups (and of the related dual pairs) can be traced back from [7, 10, 19]. Here we aim at completing the task, undertaken in [2] of (restoring and) displaying the relationship between the two developments.

Before formulating the main result of this section we shall rewrite the CR (3.13), (3.14) in terms of the discrete modes of W, A and A^* and introduce along the way the conformal Hamiltonian. We first list the $u(\infty, \infty)$ modes of W [2] and write down their CR. Here belong the generators $E^{\epsilon}_{ij}(\epsilon = +, -)$ of the maximal compact subalgebra $u(\infty) \oplus u(\infty)$ of $u(\infty, \infty)$ and of the noncompact raising and lowering operators X_{ij} and X^*_{ij}, respectively $(i, j = 1, 2, \ldots)$ satisfying

$$\begin{aligned}
&\left[E^+_{ij}, E^+_{kl}\right] = \delta_{jk}E^+_{il} - \delta_{il}E^+_{kj}, \qquad \left[E^-_{ij}, E^-_{kl}\right] = \delta_{jk}E^-_{il} - \delta_{il}E^-_{kj}, \qquad \left[E^+_{ij}, E^-_{kl}\right] = 0,\\
&\left[E^+_{ij}, X^*_{kl}\right] = \delta_{jl}X^*_{ki}, \qquad \left[E^+_{ij}, X_{kl}\right] = -\delta_{il}X_{kj},\\
&\left[E^-_{ij}, X^*_{kl}\right] = \delta_{jk}X^*_{il}, \qquad \left[E^-_{ij}, X_{kl}\right] = -\delta_{ik}X_{jl},\\
&[X_{ij}, X^*_{kl}] = \delta_{ik}E^+_{lj} + \delta_{jl}E^-_{ki}.
\end{aligned} \tag{4.1}$$

The commuting diagonal elements E^{ϵ}_{ii} span a compact *Cartan subalgebra*. The *antisymmetric bifield* A gives rise to an *Abelian algebra* spanned by the *raising operators* $Y^+_{ij} = -Y^+_{ji}$, the *lowering operators* $\left(Y^-_{ij}\right)^* = -\left(Y^-_{ji}\right)^*$ and the operators F_{ij}; the modes of A^* are Hermitian conjugate to those of A. The above E's together with F_{ij} and their conjugates, F^*_{ij}, give rise to the maximal compact subalgebra $u(2\infty)$ of $so^*(4\infty)$. The additional nontrivial CR can be restored (applying when necessary Hermitian conjugation) from the following ones:

$$\begin{aligned}
&\left[E^-_{ij}, F_{kl}\right] = \delta_{jk}F_{il}, \qquad \left[F_{ij}, E^+_{kl}\right] = \delta_{jk}F_{il}, \qquad \left[F_{ij}, F^*_{kl}\right] = \delta_{jl}E^-_{ik} - \delta_{ik}E^+_{lj};\\
&\left[X_{ij}, F_{kl}\right] = \delta_{ik}Y^+_{jl}, \qquad \left[X_{ij}, F^*_{kl}\right] = -\delta_{jl}Y^-_{ik},\\
&\left[Y^{\epsilon}_{ij}, E^{\epsilon}_{kl}\right] = \delta_{jk}Y^{\epsilon}_{il} - \delta_{ik}Y^{\epsilon}_{jl},\\
&\left[Y^+_{ij}, X^*_{kl}\right] = \delta_{il}F_{kj} - \delta_{jl}F_{ki},\\
&\left[Y^-_{ij}, X^*_{kl}\right] = \delta_{jk}F^*_{il} - \delta_{ik}F^*_{jl};\\
&\left[Y^{\epsilon}_{ij}, \left(Y^{\epsilon}_{kl}\right)^*\right] = \delta_{ik}E^{\epsilon}_{lj} - \delta_{jk}E^{\epsilon}_{li} + \delta_{jl}E^{\epsilon}_{ki} - \delta_{il}E^{\epsilon}_{kj};\\
&\left[Y^+_{ij}, F^*_{kl}\right] = \delta_{il}X_{kj} - \delta_{jl}X_{ki},\\
&\left[Y^-_{ij}, F_{kl}\right] = \delta_{jk}X_{il} - \delta_{ik}X_{jl}.
\end{aligned} \tag{4.2}$$

We note that the CR (4.1) and (4.2) do not depend on the 'central charge' $2N$ of the inhomogeneous terms in equations (3.13) and (3.14) that is absorbed in the definition of

E^{ϵ}_{ii} (cf equation (A.4) of appendix). The parameter N reappears, however, in the expression for the *conformal Hamiltonian* H_c which involves an infinite sum of Cartan modes—and hence only belongs to the completion of $u(\infty,\infty) \subset so^*(4\infty)$, which makes the central extension nontrivial

$$H_c = \sum_{i=1}^{\infty} \varepsilon_i \left(E^+_{ii} + E^-_{ii} - 2N\right). \tag{4.3}$$

Here the energy eigenvalues ε_i form an increasing sequence of positive integers (in $D = 4$: $\varepsilon_1 = 1, \varepsilon_2 = \cdots = \varepsilon_5 = 2, \varepsilon_6 = \cdots = \varepsilon_{14} = 3$, etc). The *charge* Q and the *number operator* C^u_1 which span the centre of $u(\infty,\infty)$ and of $u(2\infty)$, respectively, also involve infinite sums of Cartan modes

$$Q = \sum_{i=1}^{\infty} \left(E^+_{ii} - E^-_{ii}\right), \qquad C^u_1 = \sum_{i=1}^{\infty} \left(E^+_{ii} + E^-_{ii} - 2N\right). \tag{4.4}$$

$A\ priori\ N$ is a (positive) real number. It has been proven in [16, 15], however, that in a unitary positive-energy realization of any algebra of bifields generated by local scalar fields of scaling dimension two, N must be a natural number.

Let us define the *vacuum representation* of the bifields W and $A^{(*)}$ obeying the CR (3.13) and (3.14) as the *unitary irreducible positive-energy representation* (UIPER) of $so^*(4\infty)$ in which H_c is well defined and has eigenvalue zero on the ground state $|vac\rangle$ (the *vacuum state*). We are now ready to state our main result.

Theorem 1. *In any UIPER (of fixed N) of $so^*(4\infty)$ we have:*

(i) N is a nonnegative integer and all UIPERs of $so^*(4\infty)$ are realized (with multiplicities) in the Fock space $\mathcal{F}_{2N}$ of $2N$ free complex massless scalar fields (see the appendix).
(ii) The ground states of equivalent UIPERs of $so^*(4\infty)$ in $\mathcal{F}_{2N}$ form irreducible representations of the gauge group $Sp(2N)$. This establishes a one-to-one correspondence between UIPERs of $so^*(4\infty)$ occurring in the Fock space and the irreducible representations of $Sp(2N)$.

The *proof* parallels that of theorem 1 in [2, section 2] using the results of appendix. We shall only note that each UIPER of $so^*(4\infty)$ is expressed in terms of the fundamental weights Λ_ν of $so^*(4n)$ (for large enough n, exceeding N)

$$\Lambda = \sum_{\nu=0}^{2n-1} k_\nu \Lambda_\nu, \qquad k_\nu \leqslant 0. \tag{4.5}$$

In particular, the vacuum representation has weight $-2N\Lambda_0$ (see (A.17)). Thus, each UIPER remains irreducible when restricted to some $so^*(4n)$, so that we are effectively dealing with representations of finite dimensional Lie algebras. We also note that the bifield W has a vanishing vacuum expectation value in view of (A.16), in accord with its definition as a sum of twist two local fields.

The outcome of theorem 4.1 and of theorems 1 and 3 of [2] was expected in view of the abstract results of the Doplicher–Haag–Roberts theory of superselection sectors [9, 6, 5]. However, considerable technical difficulties are encountered in relating the extension theory of bifields with the representations of the corresponding nets. Our study provides an independent derivation of DHR-type results in the field theoretic framework, advancing at the same time the program of classifying globally conformal invariant quantum field theories in four dimensions.

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Appendix. Fock space realization of the Lie algebra $so^*(4n)$ (for $n \to \infty$)

For the higher dimensional vertex algebra formalism (and the associated complex variable realization of compactified Minkowski space) used in this appendix, see [1] and references therein (for a summary, see appendix to [2]). We write the pair of vectors of complex fields (3.12) as

$$\psi(\mathrm{z}) = a(\mathrm{z}) + b^*(\mathrm{z}), \qquad \psi^*(\mathrm{z}) = a^*(\mathrm{z}) + b(\mathrm{z}), \tag{A.1}$$

where $\psi = (\vec{\psi}_\alpha : \alpha = 1, 2) = \left(\psi_\alpha^p : \alpha = 1, 2,\ p = 1, \ldots, N\right)$ and likewise for a, b. Their mode decomposition in the compact picture is

$$\begin{aligned} \vec{a}_\alpha(\mathrm{z}) &= \sum_{\ell=0}^{\infty} \frac{1}{\sqrt{\ell+1}} \sum_{\mu=1}^{(\ell+1)^2} \frac{\vec{a}_{\alpha n}}{(\mathrm{z}^2)^{\ell+1}} h_{\ell,\mu}(\mathrm{z}), \\ \vec{b}^*_\beta(\mathrm{z}) &= \sum_{\ell=0}^{\infty} \frac{1}{\sqrt{\ell+1}} \sum_{\mu=1}^{(\ell+1)^2} \vec{b}_{\beta n} h_{\ell,\mu}(\mathrm{z}), \end{aligned} \tag{A.2}$$

where $\{h_{\ell,\mu}(\mathrm{z}) : \mu = 1, \ldots, (\ell+1)^2\}$ is a basis of homogeneous harmonic polynomials of degree ℓ in the four-vector z, diagonalizing the conformal one-particle energy, $n = n(\ell, \mu)$ $(= 1, 2, \ldots)$ is an enumeration, and $a_n^{(*)}, b_n^{(*)}$ obey the canonical commutation relations (we only list the nontrivial ones)

$$\left[a_{\alpha m}^p, a_{\beta n}^{q*}\right] = \delta_{\alpha\beta}\delta_{mn}\delta^{pq} = \left[b_{\alpha m}^p, b_{\beta n}^{q*}\right]. \tag{A.3}$$

The corresponding modes of the bifield W from (3.11) (i.e. the $u(\infty, \infty)$ generators) are split into two groups. First, we have the compact $(u(\infty) \oplus u(\infty))$ generators

$$E_{ij}^+ = \tfrac{1}{2}[a_i^*, a_j]_+ = a_i^* a_j + N\delta_{ij}, \qquad E_{ij}^- = \tfrac{1}{2}[b_i^*, b_j]_+ = b_i^* b_j + N\delta_{ij}, \tag{A.4}$$

where $a_i^* a_j$, etc stand for the inner products

$$a_i^* a_j = \sum_{\alpha=1}^{2} \vec{a}_{\alpha i}^* \cdot \vec{a}_{\alpha j} = \sum_{\alpha=1}^{2} \sum_{p=1}^{N} a_{\alpha i}^{p*} a_{\alpha j}^p. \tag{A.5}$$

Second, we have the energy decreasing (X_{ij}) and energy increasing (X_{ij}^*) operators:

$$X_{ij} = b_i a_j \left(\equiv \sum_{\alpha=1}^{2} \vec{b}_{\alpha i} \cdot \vec{a}_{\alpha j} \right), \qquad X_{ij}^* = b_i^* a_j^*. \tag{A.6}$$

The modes of the skewsymmetric bifield A from (3.11) and its conjugate also include a compact part $\left(F_{ij}^{(*)}\right)$ and a noncompact one $\left(Y_{ij}^{\pm(*)}\right)$

$$
\begin{aligned}
F_{ij} &= \vec{b}_{1i}^* \cdot \vec{a}_{2j} - \vec{b}_{2i}^* \cdot \vec{a}_{1j}; \\
Y_{ij}^+ &= \vec{a}_{1i}^* \cdot \vec{a}_{2j} - \vec{a}_{2i}^* \cdot \vec{a}_{1j} \left(= -Y_{ji}^+ \right),
\end{aligned}
\tag{A.7}
$$

$$
Y_{ij}^- = \vec{b}_{1i}^* \cdot \vec{b}_{2j} - \vec{b}_{2i}^* \cdot \vec{b}_{1j} \left(= -Y_{ji}^- \right) \tag{A.8}
$$

and their conjugates.

We shall now present the subalgebras obtained by restricting the modes to the n lowest one-particle energies. Because we wish to treat positive-energy representations as highest-weight representations, it is convenient to assign positive roots to energy lowering operators. According to the ordering of energies $\varepsilon_1^- = \varepsilon_1^+ \leqslant \varepsilon_2^- = \varepsilon_2^+ \leqslant \cdots$ (dealing with degeneracies as in [2]) we choose the (ordered set of) simple roots and raising operators (= energy lowering operators) as

$$
\begin{array}{lll}
\alpha_0 = -e_1 - e_2, & H_0 = -E_{11}^- - E_{11}^+, & X_{11} (\equiv E_0), \\
\alpha_1 = e_1 - e_2, & H_1 = E_{11}^- - E_{11}^+, & F_{11} (\equiv E_1), \\
\alpha_2 = e_2 - e_3, & H_2 = E_{11}^+ - E_{22}^-, & F_{12}^* (\equiv E_2), \\
\ldots & \ldots & \ldots \\
\alpha_{2n-2} = e_{2n-2} - e_{2n-1}, & H_{2n-2} = E_{n-1n-1}^+ - E_{nn}^-, & F_{n-1n}^* (\equiv E_{2n-2}), \\
\alpha_{2n-1} = e_{2n-1} - e_{2n}, & H_{2n-1} = E_{nn}^- - E_{nn}^+, & F_{nn} (\equiv E_{2n-1}).
\end{array}
\tag{A.9}
$$

Here the names H_ν and E_ν of the generators comply with the standard Chevalley–Serre notation; the vectors $\{e_s\}$ form an orthonormal basis so that the scalar products $(\alpha_i|\alpha_j)$ reproduce the Cartan matrix of $so^*(4n)$

$$
\begin{aligned}
&(\alpha_i|\alpha_i) = 2, \qquad (\alpha_0|\alpha_1) = 0, \\
&(\alpha_0|\alpha_2) = (\alpha_1|\alpha_2) = -1 = (\alpha_i|\alpha_{i+1})
\end{aligned}
\tag{A.10}
$$

for $i = 2, \ldots, 2n-2$. The positive roots (corresponding to the raising operators) are $e_i - e_j$ and $-e_i - e_j$ $(1 \leqslant i < j \leqslant 2n)$.

The sum t of the vectors e_s is a root vector, the corresponding Cartan element H_t generating the center of the maximal compact Lie subalgebra $u(2n)$ of $so^*(4n)$

$$
t = \sum_{s=1}^{2n} e_s, \qquad H_t = \sum_{i=1}^{n} \left(E_{ii}^+ + E_{ii}^- \right). \tag{A.11}
$$

The $so^*(4n)$ fundamental weights Λ_ν and the half sum δ of positive roots of $so^*(4n)$ are given by

$$
\begin{aligned}
&\Lambda_0 = -\frac{t}{2}, \qquad \Lambda_1 = e_1 - \frac{t}{2}, \\
&\Lambda_j = \sum_{s=1}^{j} e_s - t = -\sum_{s=j+1}^{2n} e_s \qquad (j = 2, \ldots, 2n-1),
\end{aligned}
\tag{A.12}
$$

$$
\delta = \sum_{\nu=0}^{2n-1} \Lambda_\nu = -\sum_{s=1}^{2n} (s-1) e_s = \rho - \left(n - \frac{1}{2} \right) t, \tag{A.13}
$$

where ρ is the half sum of positive roots of $su(2n)$

$$
\rho = \sum_{s=1}^{n} \left(n - s + \frac{1}{2} \right) (e_s - e_{2n+1-s}). \tag{A.14}
$$

Note that $(t|\alpha) = 0 = (t|\rho)$ for α a root of $su(2n)$; observe that ρ, δ and the Casimir invariants below are only defined for finite n. The second-order Casimir operators of $so^*(4n) \supset u(2n) \supset su(2n)$ are related by

$$C_2^{u(2n)} = C_2^{su(2n)} + \frac{H_t^2}{2n} = C_2^{so^*(4n)} + 2\sum_{j=1}^{n} X_{ij}^* X_{ij}$$
$$+2 \sum_{1\leqslant i<j\leqslant n} \left(Y_{ij}^{+*} Y_{ij}^{+} + Y_{ij}^{-*} Y_{ij}^{-}\right) + (2n-1)H_t. \tag{A.15}$$

The *vacuum* $|vac\rangle$ is defined as a basis vector in a one-dimensional space satisfying the relations $\vec{a}_{\alpha i}|vac\rangle = 0 = \vec{b}_{\alpha i}|vac\rangle$, or equivalently

$$X_{ij}|vac\rangle = Y_{ij}^{\pm}|vac\rangle = 0 = F_{ij}^{(*)}|vac\rangle, \qquad E_{ij}^{\pm}|vac\rangle = N\delta_{ij}|vac\rangle. \tag{A.16}$$

It follows that it can be identified with the highest weight vector of a unitary irreducible representation of $so^*(4n)$ (for any $n > 1$) of weight $-2N\Lambda_0$

$$|vac\rangle = |-2N\Lambda_0\rangle, \qquad C_2^{so^*(4n)}(-2N\Lambda_0) = 2nN(N+1-2n). \tag{A.17}$$

As anticipated by the ordering (A.9) of roots (and of Cartan and raising operators), it is convenient to relabel the oscillators by setting

$$\vec{A}_{2i-1} = -\vec{b}_{2,i}, \qquad \vec{A}_{2i} = \vec{a}_{1,i}, \qquad \vec{B}_{2i-1} = \vec{b}_{1,i}, \qquad \vec{B}_{2i} = \vec{a}_{2,i}, \qquad i = 1,\ldots,n. \tag{A.18}$$

Then the generators of $so^*(4n)$ can be rewritten as

$$\begin{aligned} &E^{\pm}, F, F^* \to E_{kl} = \vec{A}_k^* \cdot \vec{A}_l + \vec{B}_k^* \cdot \vec{B}_l + \delta_{kl}N, \\ &X, Y^{\pm} \to Y_{kl} = \vec{A}_k \cdot \vec{B}_l - \vec{B}_k \cdot \vec{A}_l \qquad (k,l = 1,\ldots 2n). \end{aligned} \tag{A.19}$$

We refrain from displaying the commutation relations of $so^*(4n)$ again, which are most easily (and more compactly than (4.2)) read off this representation.

Our aim is to classify the UIPERs of $so^*(4n)$ with ground states $|h\rangle$ with Cartan eigenvalues

$$E_{kk}|h\rangle = h_k|h\rangle. \tag{A.20}$$

We omit the details of the argument, which is in perfect analogy with [2], indicating only the three main steps.

1. Unitarity of the submodule $U(h)$ obtained by acting with the generators of the maximal compact Lie subalgebra $u(2n)$ on the ground state implies that

$$h_1 \geqslant h_2 \geqslant h_3 \geqslant \cdots \tag{A.21}$$

is an integer-spaced non-increasing sequence, stabilizing at some value h_∞, and $h_\infty = 2N/2 = N$ in order to have a finite Hamiltonian. The finiteness of the operators Q and C_1^u on all states of finite energy is then automatically guaranteed.

2. We choose n large enough so that $h_{2n} = h_\infty = N$. Let $\mathscr{Y}$ be the Young tableau of $su(2n)$ with rows of length $m_k = h_k - h_\infty$.

The noncompact generators Y_{kl}^* with negative roots transform like the antisymmetric rank 2 representation of $u(2n)$. Hence, the linear span of $Y^*U(h)$ decomposes into irreducible representations of $u(2n)$ whose Young diagrams are obtained by adding two boxes in different rows to $\mathscr{Y}$. Their highest weights λ are of the form $h + e_k + e_l$, where $k \neq l$.

In each of these states, the above Casimir operators can be computed. Since the difference $C_2^{u(2n)} - C_2^{so^*(4n)}$ is a positive operator, the difference of eigenvalues must be nonnegative. This yields the necessary bounds

$$(\lambda+\delta, \lambda+\delta) - (h+\delta, h+\delta) \geqslant 0 \tag{A.22}$$

for all $\lambda = h + e_k + e_l$. The strongest bound occurs when k and l are chosen maximal, i.e. $k = r + 1$ and $l = r + 2$ when $r + 1$ is the smallest index such that $h_{r+1} = h_\infty$ (i.e. r is the number of the rows of the Young diagram $\mathscr{Y}$). Evaluating the bound, yields the condition

$$r \leqslant N. \tag{A.23}$$

3. The Young diagrams admitted by this condition are precisely those of the unitary tensor representations of $U(N)$. It remains to establish the relation between these and the unitary representations of the gauge group $Sp(2N)$ of the field algebra (3.13) (which contains $U(N)$), and to verify that each of these is realized on the Fock space of $2N$ complex massless free fields ψ^p_α ($\alpha = 1, 2$, $p = 1, \ldots, N$).

By the above relabeling of the oscillators, the infinitesimal generators of $sp(2N)$ become

$$E^{pq} = E^{qp*} = \sum_{k=1}^{2\infty} \left(A^{p*}_k A^q_k - B^{q*}_k B^p_k\right), \tag{A.24}$$

$$X^{pq} = X^{qp} = \sum_{k=1}^{2\infty} \left(A^{p*}_k B^q_k + A^{q*}_k B^p_k\right), \qquad p, q = 1, \ldots, N. \tag{A.25}$$

The E^{pq} are the generators of $U(N) \subset Sp(2N)$, and A^* (the creation operators for $\vec{\psi}^*_1$ and for $\vec{\psi}_2$) transform in the vector representation of $U(N)$, while B^* (the creation operators for $\vec{\psi}^*_2$ and for $\vec{\psi}_1$) transform in the conjugate representation. In other words, one may assign the weights e^p to A^{p*} and $-e^p$ to B^{p*}, so that E^{pq} correspond to the roots $e^p - e^q$ and X^{pq} to $-e^p - e^q$. The simple roots are $e^p - e^{p+1}$ (corresponding to $SU(N) \subset Sp(2N)$) and $2e^N$.

Now let $(h_1, h_2, \ldots, h_N, h_{N+1} = \cdots = h_n = N)$ be the Cartan weights of a positive-energy representation of $so^*(4n) \subset so^*(4\infty)$. Let $\mathscr{Y}$ be the Young diagram of $U(N)$ with rows of length $m_k = h_k - N$, and r_l the heights of its columns.

Define in the Fock space of the complex free fields $\vec{\psi}^{(*)}_1$ and $\vec{\psi}^{(*)}_2$ the vector

$$|h\rangle_F = \left(\prod_{l=1}^{m_1} A^{*\wedge r_l}\right) |vac\rangle, \tag{A.26}$$

where $A^{*\wedge r} = \det\left(A^{p*}_k\right)^{p=1,\ldots r}_{k=1,\ldots r}$. Then $|h\rangle_F$ is a highest weight vector for $so^*(4\infty)$ with the proper Cartan eigenvalues h_k of E_{kk}. It is a component of a $U(N)$ tensor in the representation given by $\mathscr{Y}$. This tensor extends, by the action of the generators X^{pq}, X^{pq*}, to a $Sp(2N)$ tensor. (The generators X^{pq*} will swap some of the A-excitations into B-excitations.)

As a representation of $u(N)$, this representation has highest weight $w = m_1 e^1 + \cdots + m_N e^N$. We decompose this into the fundamental weights of $sp(2N)$. These are determined by the property that $(\Lambda^l, \alpha_k) = \delta^l_k$ where α_k are the simple roots, giving $\Lambda^l = e^1 + \cdots + e^l$ ($l = 1, \ldots N - 1$) and $\Lambda^N = \frac{1}{2}\sum_{p=1}^N e^p$. Then

$$w = n_1 \Lambda^1 + \cdots + n_N \Lambda^N \tag{A.27}$$

with $n_l = m_l - m_{l+1}$ ($l < N$), and $n_N = 2m_N$. We therefore obtain all those representations of $sp(2N)$ for which n_N is even.

Representations with half-integral weights (n_N odd) integrate to representations of a two-fold covering of $Sp(2N)$, because the $U(1)$ subgroups $\exp itE^{pp}$ integrate to -1 as $t = 2\pi$. Thus, we obtain the desired duality result: all irreducible positive-energy representations of $su^*(4\infty)$ are realized on the Fock space, and their multiplicity spaces are representation spaces of all irreducible unitary true representations of the gauge group $Sp(2N)$.

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